

THE PRIMITIVE SOLUTIONS TO $x^3 + y^9 = z^2$

NILS BRUIN

ABSTRACT. We determine the rational integers x, y, z such that $x^3 + y^9 = z^2$ and $\gcd(x, y, z) = 1$. First we determine a finite set of curves of genus 10 such that any primitive solution to $x^3 + y^9 = z^2$ corresponds to a rational point on one of those curves. We observe that each of these genus 10 curves covers an elliptic curve over some extension of \mathbb{Q} . We use this cover to apply a Chabauty-like method to an embedding of the curve in the Weil restriction of the elliptic curve. This enables us to find all rational points and therefore deduce the primitive solutions to the original equation.

1. INTRODUCTION

In this article, we consider a special instance of the equation $Ax^r + By^s = Cz^t$. One of the most important results for this equation is a theorem by Darmon and Granville ([7]), which for fixed, nonzero A, B, C and fixed r, s, t satisfying $1/r + 1/s + 1/t < 1$, relates primitive solutions x, y, z (solutions with $x, y, z \in \mathbb{Z}$ and $\gcd(x, y, z) = 1$) to rational points on finitely many algebraic curves of general type. It follows by Faltings' theorem that only finitely many primitive solutions exist. Unfortunately, their proof does not provide a recipe to produce the relevant curves.

As Tijdeman shows in [12], the ABC-conjecture implies that if one leaves A, B, C fixed, but allows r, s, t to vary subject to $1/r + 1/s + 1/t < 1$, then the number for primitive solutions to $Ax^r + By^s = Cz^t$ is still finite.

The case $A = B = C = 1$ has received most attention. An extensive computer search performed by Beukers and Zagier (see [1]) showed there are some surprisingly large primitive solutions to $x^r + y^s = z^t$ for

$$(1) \quad \{r, s, t\} \in \{\{2, 3, 7\}, \{2, 3, 8\}, \{2, 3, 9\}, \{2, 4, 5\}\}.$$

We call a solution *trivial* if one of x, y, z is equal to ± 1 . Note that this includes $2^3 + 1^r = 3^2$ and, for primitive solutions, also any solution satisfying $xyz = 0$. With this definition, the only equations $x^r + y^s = z^t$ with $1/r + 1/s + 1/t < 1$ for which non-trivial primitive solutions are known, are the ones satisfying (1).

It was noted by Tijdeman and Zagier that the known non-trivial primitive solutions of $x^r + y^s = z^t$ all have $\min(r, s, t) \leq 2$. They conjectured that for $r, s, t \geq 3$ only trivial primitive solutions exist. The resolution of this conjecture has even attracted a monetary prize (see [9]).

Date: October 31, 2003.

1991 Mathematics Subject Classification. Primary 11D41; Secondary 11G30.

Key words and phrases. Generalised Fermat equation, Chabauty methods, rational points on curves, covering techniques.

The author is partially supported by NSERC and the research described in this paper was done partially at the University of Sydney.

In this paper, we work in a different direction. Because the equations satisfying (1) have known non-trivial primitive solutions, the question whether they have any others appears natural. In his thesis ([4], [5] and [3]), the author resolved the cases $\{r, s, t\} \in \{\{2, 3, 8\}, \{2, 4, 5\}\}$. Recently Poonen, Schaefer and Stoll have made excellent progress on $x^2 + y^3 = z^7$. In this paper we deal with the one remaining case:

Theorem 1.1. *The primitive solutions to the equation $x^3 + y^9 = z^2$ are*

$$(x, y, z) \in \{(1, 1, 0), (0, 1, \pm 1), (1, 0, \pm 1), (2, 1, \pm 3), (-7, 2, \pm 13)\}.$$

2. PARAMETRISING CURVES FOR $x^3 + y^9 = z^2$

We note that the perfect 9th powers form a subset of the perfect cubes, so any primitive solution (x, y, z) to $x^3 + y^9 = z^2$ gives rise to a primitive solution (x, v, z) of $x^3 + v^3 = z^2$ by putting $v = y^3$. The solutions of the latter equation can be classified by the following theorem.

Theorem 2.1 (Mordell, [10, Chapter 25]). *If $x, v, z \in \mathbb{Z}$ with $x^3 + v^3 = z^2$ and $\gcd(x, v, z) = 1$, then there are $s, t \in \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ and $(s, t) \not\equiv (0, 0) \pmod{p}$ for any prime $p \nmid 6$, such one of the following holds.*

$$\begin{aligned} x \text{ or } v &= s^4 + 6s^2t^2 - 3t^4 \\ v \text{ or } x &= -s^4 + 6s^2t^2 + 3t^4 \\ z &= \pm 6st(s^4 + 3t^4) \\ x \text{ or } v &= \frac{1}{4}(s^4 + 6s^2t^2 - 3t^4) \\ v \text{ or } x &= \frac{1}{4}(-s^4 + 6s^2t^2 + 3t^4) \\ z &= \pm \frac{3}{4}st(s^4 + 3t^4) \\ x \text{ or } v &= s(s^3 + 8t^3) \\ v \text{ or } x &= 4t(t^3 - s^3) \\ z &= \pm s^6 - 20s^3t^3 - 8t^6 \end{aligned}$$

It follows that in order to find all primitive solutions to $x^3 + y^9 = z^2$, it suffices to find all solutions y, s, t with $y \in \mathbb{Z}$ and $s, t \in \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ with s, t not both divisible by any prime p not dividing 6, to the following equations:

$$\begin{aligned} 1: \quad y^3 &= s^4 + 6s^2t^2 - 3t^4 \\ 2: \quad y^3 &= -s^4 + 6s^2t^2 + 3t^4 \\ 3: \quad y^3 &= \frac{1}{4}(s^4 + 6s^2t^2 - 3t^4) \\ 4: \quad y^3 &= \frac{1}{4}(-s^4 + 6s^2t^2 + 3t^4) \\ 5: \quad y^3 &= s(s^3 + 8t^3) \\ 6: \quad y^3 &= 4t(t^3 - s^3). \end{aligned}$$

This leads us to a generalisation of the concept of *primitive solution*. Let $S = \{p_1, \dots, p_s\}$ be a finite set of primes. We write

$$\mathbb{Z}_S = \mathbb{Z}[1/p_1, \dots, 1/p_s].$$

A tuple $(x_1, \dots, x_n) \in \mathbb{Z}_S$ is called *S-primitive* if the ideal $(x_1, \dots, x_n)\mathbb{Z}_S$ equals \mathbb{Z}_S . Equivalently, it means that no prime outside S divides all x_i . In order to determine the primitive solutions of $x^3 + y^9 = z^2$, it suffices to determine the *S-primitive* solutions to the equations above.

Obviously, if (s, t, y) is a solution to one of these equations, then so is an entire class of weighted homogeneously equivalent solutions of the form $(\lambda^3 s, \lambda^3 t, \lambda^4 y)$.

Furthermore, a solution (s, t, y) to one of the equations 1 or 2 gives rise to a solution $(\frac{1}{2}s, \frac{1}{2}t, \frac{1}{4}y)$ of equations 3 or 4 respectively. Note that the quantity $(s : t)$ is invariant under either transformation. Since a solution can be easily reconstructed (up to equivalence), it is sufficient to determine the values of s/t that can occur for $\{2, 3\}$ -primitive solutions of the equations 1, 2, 5 and 6.

Following [7], the equivalence classes under homogeneous equivalence of $\{2, 3\}$ -primitive solutions to each of these equations correspond to the rational points on a finite collection of algebraic curves. We construct these curves explicitly (see also [5]). We introduce some notation.

Let K be a number field and let S be a finite set of rational primes. For a prime \mathfrak{p} of K we write $\mathfrak{p} \nmid S$ if $v_{\mathfrak{p}}(q) = 0$ for all $q \in S$. Following [11], for any rational prime p we define the following subgroup of K^*/K^{*p} :

$$K(p, S) = \{a \in K^* : v_{\mathfrak{p}}(a) \equiv 0 \pmod{p} \text{ for all primes } \mathfrak{p} \nmid S\} / K^{*p}.$$

This is a finite group. We will identify it with a set of representatives in K^* .

Let $f \in K[x]$ be a square-free polynomial and let $A := K[x]/(f)$. The algebra A is isomorphic to a direct sum of number fields K_1, \dots, K_r and $A^* = K_1^* \times \dots \times K_r^*$. We generalise the notation above by defining

$$A(p, S) := K_1(p, S) \times \dots \times K_r(p, S) \subset A^*/A^{*p}.$$

We identify the elements of this finite group with a set of representatives in A^* .

Lemma 2.2. *Let $f(s, t) \in \mathbb{Z}[s, t]$ be a square-free homogeneous form of degree 4. Let $S = \{p \text{ prime} : p \mid 3\text{Disc}(f)\}$. Solutions of $y^3 = f(s, t)$ with $y, s, t \in \mathbb{Q}$, s, t integral outside S and $(s, t) \pmod{p} \neq (0, 0)$ for any $p \notin S$ correspond, up to weighted projective equivalence, to rational points on finitely many explicitly constructible smooth projective curves of the form*

$$\begin{aligned} Q_{2,\delta}(y_0, y_1, y_2, y_3) &= 0 \\ Q_{3,\delta}(y_0, y_1, y_2, y_3) &= 0, \end{aligned}$$

indexed by $\delta \in A(3, S)$ for some explicit semisimple algebra A . For each of these curves, the ratio s/t can be explicitly expressed as a function

$$\frac{s}{t} = -\frac{Q_{0,\delta}(y_0, y_1, y_2, y_3)}{Q_{1,\delta}(y_0, y_1, y_2, y_3)}.$$

Proof. Using an $\text{SL}_2(\mathbb{Z})$ transformation on s, t if necessary, we can assume that f is of degree 4 in s . We form the algebra $A = \mathbb{Q}[\theta] = \mathbb{Q}[x]/f(x, 1)$. Let C be the leading coefficient of $f(x, 1)$. Then we have $f = C N_{A[s,t]/\mathbb{Q}[s,t]}(s - \theta t)$, so having a solution would amount to the existence of $y_0, y_1, y_2, y_3 \in \mathbb{Q}$ and $\delta \in A^*$ such that

$$\begin{aligned} (s - \theta t) &= \delta(y_0 + \theta y_1 + \theta^2 y_2 + \theta^3 y_3)^3 \\ y &= \sqrt[3]{C N_{A/\mathbb{Q}}(\delta) N_{A[s,t]/\mathbb{Q}[s,t]}(y_0 + \theta y_1 + \theta^2 y_2 + \theta^3 y_3)}. \end{aligned}$$

It follows immediately that $C N_{A/\mathbb{Q}}(\delta)$ should be a perfect cube and that the value of δ is only relevant modulo cubes.

The fact that s, t are coprime implies that δ can be chosen from a finite set of representatives. From, for instance, [4, Lemma 2.2.1] it follows that if s, t are integral and coprime outside of S , then $\delta \in A(3, S)$.

Given $\delta \in A(3, S)$ with $N(\delta) \in \mathbb{Q}^{*3}$, there are unique cubic forms $Q_{0,\delta}, \dots, Q_{3,\delta} \in \mathbb{Q}[y_0, \dots, y_3]$ such that

$$\delta(y_0 + \theta y_1 + \theta^2 y_2 + \theta^3 y_3)^3 = Q_{0,\delta} + Q_{1,\delta}\theta + Q_{2,\delta}\theta^2 + Q_{3,\delta}\theta^3$$

Solving s, t from the equations above, we obtain

$$\begin{aligned} s &= Q_{0,\delta}(y_0, y_1, y_2, y_3) \\ t &= -Q_{1,\delta}(y_0, y_1, y_2, y_3) \\ 0 &= Q_{2,\delta}(y_0, y_1, y_2, y_3) \\ 0 &= Q_{3,\delta}(y_0, y_1, y_2, y_3) \end{aligned}$$

The latter two equations define a smooth projective curve of genus 10 (see [4]) and the first two equations show that s/t is a rational function on that curve. \square

3. RATIONAL POINTS ON THE PARAMETRISING CURVES

We now proceed to apply this procedure to each of the equations mentioned above and then determine the finite set of values that s/t attains for the $\{2, 3\}$ -primitive solutions. Lemmas 3.1 and 3.3 both require standard but elaborate and tedious computations. These are easily executed by a computer algebra system, but are too bulky to reproduce completely on paper. We only give the basic data to check these computations. For the interested reader, we have made available a complete electronic transcript [2] of the computations in MAGMA [8].

Lemma 3.1. *The $\{2, 3\}$ -primitive solutions of the equation*

$$y^3 = s(s^3 + 8t^3)$$

have

$$\frac{s}{t} \in \{-2, 0, 1, 2, 4, \infty\}.$$

Proof. We use the construction in the proof of Lemma 2.2. In our case, $A = \mathbb{Q}[\theta]$, where $\theta(\theta+2)(\theta^2-2\theta+4) = 0$ and $S = \{2, 3\}$. The subgroup of $A(3, S)$ of elements of cubic norm is spanned by

$$\begin{aligned} &-1/8\theta^3 - 1/4\theta^2 - \theta + 1, \\ &-1/24\theta^3 - 1/6\theta^2 - 2/3\theta + 1, \\ &1/24\theta^3 + 1/6\theta^2 + 1/6\theta + 1, \\ &5/24\theta^3 - 5/12\theta^2 - 2/3\theta + 3, \\ &1/8\theta^3 - 1/4\theta^2 - 1/2\theta + 2. \end{aligned}$$

For each of the 3^5 possible values of δ in that group, we can write down the curve $Q_{2,\delta} = Q_{3,\delta} = 0$ as outlined in the proof of Lemma 2.2 and test the curve for local solubility over \mathbb{Q}_3 . Only 22 values for δ pass this test.

From the factorisation of f it follows that there are two homomorphisms $m_1, m_2 : A \rightarrow \mathbb{Q}$, defined by $m_1(\theta) = 0$ and $m_2(\theta) = -2$. For a fixed value of δ , the curve $Q_{2,\delta} = Q_{3,\delta} = 0$ covers two curves of genus 1:

$$\begin{aligned} E_{1,\delta} &: \frac{N(\delta)}{m_1(\delta)} u_1^3 = s^3 - 8t^3 \\ E_{2,\delta} &: \frac{N(\delta)}{m_2(\delta)} u_2^3 = s(s^2 - 2st + 4t^2) \end{aligned}$$

Note that $(s : t : u_1) = (2 : 1 : 0) \in E_{1,\delta}(\mathbb{Q})$ and $(s, t, u_2) = (0 : 1 : 0) \in E_{2,\delta}(\mathbb{Q})$, so both curves are isomorphic to elliptic curves and hence their rational points have the structure of a finitely generated group. For the 22 values of δ , we obtain the results in Table 1. For each of the values of δ at least one of $E_{1,\delta}(\mathbb{Q})$ and $E_{2,\delta}(\mathbb{Q})$

δ	$N(\delta)/m_1(\delta)$	$N(\delta)/m_2(\delta)$	$\text{rk}E_{1,\delta}(\mathbb{Q})$	$\text{rk}E_{2,\delta}(\mathbb{Q})$
1	1	1	0	1
$-\frac{3}{8}\theta^3 + \frac{1}{2}\theta^2 - \frac{3}{2}\theta + 1$	1	3	0	0
$\frac{1}{8}\theta^3 - 3\theta + 1$	1	36	0	1
$\frac{1}{8}\theta^3 - 2\theta + 1$	1	2	0	0
$\frac{1}{24}\theta^3 + \frac{1}{6}\theta^2 + \frac{1}{6}\theta + 1$	1	1	0	1
$-\frac{3}{8}\theta^3 + \frac{3}{4}\theta^2 - \theta + 1$	1	3	0	0
$-\frac{7}{8}\theta^3 + \theta^2 - 3\theta + 1$	1	12	0	0
$\frac{1}{8}\theta^3 - 6\theta + 1$	1	18	0	1
$-\frac{1}{8}\theta^3 + \frac{1}{2}\theta^2 + \frac{1}{2}\theta + 1$	1	9	0	0
$-\frac{1}{24}\theta^3 + \frac{1}{3}\theta^2 + \frac{1}{3}\theta + 1$	1	4	0	0
$-\frac{15}{8}\theta^3 + 2\theta^2 - 6\theta + 1$	1	6	0	0
$-\frac{1}{12}\theta^3 + \frac{2}{3}\theta^2 - \frac{4}{3}\theta + 3$	9	3	1	0
$-\frac{23}{24}\theta^3 + \frac{8}{3}\theta^2 - \frac{22}{3}\theta + 3$	9	6	1	0
$\frac{1}{24}\theta^3 + \frac{11}{12}\theta^2 - \frac{4}{3}\theta + 3$	9	3	1	0
$\frac{1}{8}\theta^3 + \frac{1}{4}\theta^2 - 2\theta + 9$	3	3	0	0
$-\frac{3}{8}\theta^3 + \frac{3}{4}\theta^2 - \frac{1}{2}\theta + 2$	4	3	0	0
$\frac{7}{24}\theta^3 + \frac{5}{12}\theta^2 - \frac{11}{6}\theta + 6$	36	3	0	0
$\frac{13}{8}\theta^3 - \frac{1}{2}\theta^2 - \frac{5}{2}\theta + 18$	12	3	1	0
$\frac{1}{8}\theta^3 + \frac{3}{4}\theta^2 - \frac{7}{2}\theta + 4$	2	3	0	0
$\frac{7}{24}\theta^3 - \frac{7}{12}\theta^2 - \frac{5}{6}\theta + 4$	2	1	0	1
$\frac{23}{24}\theta^3 + \frac{1}{12}\theta^2 - \frac{13}{6}\theta + 12$	18	3	0	0
$\frac{29}{8}\theta^3 - \frac{1}{4}\theta^2 - \frac{5}{2}\theta + 36$	6	3	1	0

TABLE 1. Ranks of $E_{i,\delta}(\mathbb{Q})$ related to $y^3 = s(s^3 + 8t^3)$

is of rank 0. Hence, any $\{2, 3\}$ -primitive solution must have s/t corresponding to a torsion point on one of $E_{1,\delta}(\mathbb{Q})$ and $E_{2,\delta}(\mathbb{Q})$.

The group $E_{1,\delta}(\mathbb{Q})^{\text{tors}}$ is $\mathbb{Z}/3\mathbb{Z}$ for $N(\delta)/m_1(\delta) = 1$ and $\mathbb{Z}/2\mathbb{Z}$ for $N(\delta)/m_1(\delta) = 2$. For other values there is no torsion. The non-trivial 3-torsion points are $(s : t : u_1) = (1 : 0 : 1), (0 : 1 : 2)$. The non-trivial 2-torsion point is $(s : t : u_1) = (2 : 1 : 8)$.

The group $E_{2,\delta}(\mathbb{Q})^{\text{tors}}$ is $\mathbb{Z}/3\mathbb{Z}$ for $N(\delta)/m_1(\delta) = 3$ and $\mathbb{Z}/2\mathbb{Z}$ for $N(\delta)/m_1(\delta) = 6$. For other values there is no torsion. The non-trivial 3-torsion points are $(s : t : u_2) = (1 : 1 : 1), (2 : -1 : 8)$. The non-trivial 2-torsion point is $(s : t : u_2) = (4 : 1 : 2)$. These points give rise to the set of values stated in the lemma. \square

Lemma 3.2. *The $\{2, 3\}$ -primitive solutions of the equation*

$$y^3 = 4t(t^3 - s^3)$$

have

$$\frac{s}{t} \in \{-2, -1, -\frac{1}{2}, 0, 1, \infty\}.$$

Proof. Note that the map $(s, t, y) \mapsto (-t/2, s/4, y/4)$ is a bijection from the $\{2, 3\}$ -primitive solutions of $y^2 = s(s^3 + 8t^3)$ to those of $y^3 = 4t(t^3 - s^3)$. Lemma 3.1 together with the induced map $s/t \mapsto -2t/s$ proves the statement. \square

Lemma 3.3. *The $\{2, 3\}$ -primitive solutions of*

$$y^3 = s^4 + 6s^2t^2 - 3t^4$$

i	$E_i : y^2 = \dots$	independent points in $E_i(K)$
1	$x^3 + 2\alpha^3 - 4\alpha^2 - 2\alpha - 6$	$g_1 = (2, \alpha^3 - 2\alpha^2 - \alpha)$ $g_2 = (2\alpha^2 + 2\alpha + 2, -3\alpha^3 - 4\alpha^2 - 5\alpha - 2)$
2	$x^3 - 2\alpha^3 + 4\alpha^2 + 2\alpha - 2$	$(2\alpha, \alpha^3 - \alpha)$ $(1, \alpha^2 - 2\alpha)$
3	$x^3 + 2\alpha^3 - 6\alpha^2 + 2\alpha$	$(2, \alpha^2 - 3)$
4	$x^3 + 2\alpha^3 - 2\alpha^2 - 6\alpha$	$(2, 2\alpha^3 - 3\alpha^2 - 2\alpha - 1)$
5	$x^3 - 30\alpha^3 - 22\alpha^2 - 30\alpha$	$(2\alpha^2 + 2\alpha + 2, 6\alpha^3 - \alpha^2 + 2\alpha - 5)$
6	$x^3 + 14\alpha^3 - 6\alpha^2 - 74\alpha + 32$	$(-2\alpha^3 + 6\alpha^2 - 4\alpha + 26\alpha^3 - 13\alpha^2 - 4\alpha + 5)$

TABLE 2. Independent points on E_i over $K = \mathbb{Q}(\alpha)$

have

$$\frac{s}{t} \in \{-1, 0, 1, \infty\}$$

and the $\{2, 3\}$ -primitive solutions of

$$y^3 = -s^4 + 6s^2t^2 + 3t^4$$

have

$$\frac{s}{t} \in \{-3 - 1, 0, 1, 3, \infty\}.$$

Proof. Again we follow the procedure outlined in Lemma 2.2. For both equations, the algebra A is isomorphic to the number field $K(\alpha) := \mathbb{Q}[x]/(x^4 - 2x^3 - 2x + 1)$. In each case, we are left with 4 values of δ for which the curve $C_\delta : Q_{\delta,2} = Q_{\delta,3} = 0$ has points over \mathbb{Q}_3 . Each of those curves actually has a rational point. Over K , it covers the genus 1 curve

$$E_\delta : \frac{N(\delta)}{\delta} u^3 = \frac{N(s - \theta t)}{s - \theta t}$$

The existence of a \mathbb{Q} -rational point on C_δ implies that there is a K -rational point on E_δ , which makes E_δ isomorphic to an elliptic curve. We have the following diagram.

$$\begin{array}{ccc} C/\mathbb{Q} & \xrightarrow{\pi} & E/K \\ \downarrow \frac{s}{t} & & \uparrow \frac{s}{t} \\ \mathbb{P}^1/\mathbb{Q} & & \end{array}$$

It follows immediately that

$$\frac{s}{t}(C(\mathbb{Q})) \subset \mathbb{P}^1(\mathbb{Q}) \cap \frac{s}{t}(E(K)).$$

A method for determining the intersection on the right-hand side is described in [5]. The method is an adaptation of Chabauty's method ([6]) and applies if $\text{rk} E(K) < [K : \mathbb{Q}]$. The method requires generators of a subgroup of $E(K)$ of maximal rank. Table 2 lists such generators for the six elliptic curves that are encountered. A 2-descent shows that these points generate a subgroup of finite index prime to 2 and a simple check for 3-divisibility of all relevant linear combinations of these points that the index is also prime to 3. We will need this later on. For the first equation, using $\theta = \alpha^2 - 2\alpha$, we find that $C_\delta(\mathbb{Q}_3) \neq \emptyset$ for the 4 values of δ listed in Table 3. We also list the value of $\frac{s}{t}$ at a point $p_0 \in C_\delta(\mathbb{Q})$ and we list i such that E_δ is

δ	$\frac{s}{t}(p_0)$	i	δ	$\frac{s}{t}(p_0)$	i
1	∞	1	1	∞	2
$\alpha^3 - 2\alpha^2 - \alpha - 3$	0	2	$-4\alpha^3 + 8\alpha^2 + 4\alpha + 11$	0	1
$-3\alpha^3 + 5\alpha^2 + 2\alpha + 7$	1	3	$\alpha^2 + \alpha + 1$	1	6
$7\alpha^3 - 11\alpha^2 - 5\alpha - 16$	-1	4	$\alpha^3 + \alpha^2 + \alpha$	-1	5
$\theta = \alpha^2 - 2\alpha$			$\theta = \alpha^3 - \alpha^2 - \alpha - 2$		

TABLE 3. E_δ , $\frac{s}{t}(p_0)$ and isomorphic E_i

isomorphic to E_i in Table 2. For the second equation, using $\theta = \alpha^3 - \alpha^2 - \alpha - 2$, we find a similar set of values. Note that all the eight smooth plane cubics E_δ have a K -rational point of inflection. Thus, the isomorphism between E_δ and E_i can be realised by a linear change of variables. As an example, we give some data for $\delta = -4\alpha^3 + 8\alpha^2 + 4\alpha + 11$. The point $(u : s : t) = (0 : -\alpha^3 + \alpha^2 + \alpha + 2 : 1)$ is a point of inflection. If we map this point to the origin on E_1 , then the function $\frac{s}{t}$ on E_1 can be expressed as

$$\frac{s}{t} = \frac{(-\alpha^3 + \alpha^2 + \alpha + 2)y + (-\alpha^3 + 2\alpha^2 - \alpha + 2)}{y + (3\alpha^3 - 6\alpha^2 - 3\alpha - 8)},$$

where y is a coordinate of the Weierstrass model of E_1 as mentioned in Table 2.

We find $\frac{s}{t}(\{g_1, -g_2, -g_1 + g_2\}) = \{0, -3, 3\}$. Via a Chabauty-like argument using the primes of K over 11 (see [5]), we find that these are all points in $E_1(K)$ that have values in \mathbb{Q} under $\frac{s}{t}$.

For the other curves we can use the same method. Either the primes over 11 or over 31 yield sufficient information to conclude that there is only one point with a \mathbb{Q} -rational value of $\frac{s}{t}$. This is the value listed in the tables above. \square

4. PROOF OF THEOREM 1.1

Given Lemmas 3.1, 3.2 and 3.3, the proof of Theorem 1.1 is reduced to filling in pairs of s, t giving rise to the values $\frac{s}{t}$ listed in those lemmas in the corresponding parametrisations listed in Theorem 2.1. We then check if the resulting solution to $x^3 + v^3 = z^2$ is weighted homogeneously equivalent with a primitive solution to $x^3 + y^9 = z^2$.

Since the Lemmas 3.1 and 3.2 give values of $\frac{s}{t}$ for which the first or the second polynomial of the third parametrization in Theorem 2.1 may be a cube, we try all of the values and see what we get.

s	t	solution
-2	1	$0^3 + 36^3 = 216^2$
0	1	$0^3 + 4^3 = (-8)^2$
1	1	$9^3 + 0^3 = (-27)^2$
1	0	$1^3 + 0^3 = 1^3$
2	1	$32^3 - 28^3 = (-104)^2$
4	1	$288^3 - 252^3 = 2808^2$
-1	1	$-7^3 + 8^3 = 13^2$
-1	2	$-63^3 + 72^3 = (-351)^2$

It is easily checked that all solutions are equivalent to one occurring in the list in the theorem. Note that the first four entries correspond to values of $\frac{s}{t}$ that occur in both

Lemma 3.1 and 3.2. Correspondingly, both cubes are ninth powers, up to $\{2, 3\}$ -units. The latter four values occur in only one of the lemmas and correspondingly have primes different from 2, 3 dividing one of the cubes to a power not divisible by 9.

For the values of $\frac{s}{t}$ listed in Theorem 3.3 we proceed similarly.

s	t	solution
1	0	$1^3 - 1^3 = 0^2$
0	1	$-3^3 + 3^3 = 0^2$
± 1	1	$4^3 + 8^3 = (\pm 24)^2$
± 3	1	$132^3 - 24^3 = (\pm 1512)^2$

The last two solutions are interesting. The penultimate solution is obviously equivalent to $1^3 + 2^3 = 3^2$ and, since 1 is indeed a cube, this yields a solution. The last one is equivalent to $33^3 - 6^3 = 189^2$. Indeed, this is equivalent to the solution $(2^2 \cdot 3^3 \cdot 11)^3 - (2 \cdot 3)^9 = (2^3 \cdot 3^6 \cdot 7)^2$, which is a $\{2, 3\}$ -primitive solution. It is clear, however, that it is not equivalent to a truly primitive one.

5. ACKNOWLEDGEMENTS

I would like to thank the School of Mathematics of the University of Sydney for their generosity and stimulating environment. Furthermore, I would like to thank Don Zagier for introducing me to the subject of the paper, Bjorn Poonen for introducing me to the method of Chabauty and Victor Flynn, Ed Schaefer and Joe Wetherell for instructive comments and discussions.

REFERENCES

- [1] Frits Beukers, *The Diophantine equation $Ax^p + By^q = Cz^r$* , Duke Math. J. **91** (1998), no. 1, 61–88.
- [2] Nils Bruin, *Electronic transcript of computations*, <http://www.cecm.sfu.ca/~nbruin/eq239>.
- [3] ———, *The diophantine equations $x^2 \pm y^4 = \pm z^6$ and $x^2 + y^8 = z^3$* , Compositio Math. **118** (1999), 305–321.
- [4] ———, *Chabauty methods and covering techniques applied to generalized Fermat equations*, CWI Tract, vol. 133, Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 2002, Dissertation, University of Leiden, Leiden, 1999. MR **2003i**:11042
- [5] ———, *Chabauty methods using elliptic curves*, J. Reine Angew. Math. **562** (2003), 27–49.
- [6] Claude Chabauty, *Sur les points rationnels des variétés algébriques dont l'irrégularité est supérieure à la dimension*, C. R. Acad. Sci. Paris **212** (1941), 1022–1024.
- [7] Henri Darmon and Andrew Granville, *On the equations $z^m = F(x, y)$ and $Ax^p + By^q = Cz^r$* , Bull. London Math. Soc. **27** (1995), no. 6, 513–543.
- [8] John Cannon et al., *The Magma computational algebra system*, <http://magma.maths.usyd.edu.au>.
- [9] R. Daniel Mauldin, *A generalization of Fermat's last theorem: the Beal conjecture and prize problem*, Notices Amer. Math. Soc. **44** (1997), no. 11, 1436–1437.
- [10] L. J. Mordell, *Diophantine equations*, Pure and Applied Mathematics, Vol. 30, Academic Press, London, 1969. MR **40** #2600
- [11] Joseph H. Silverman, *The arithmetic of elliptic curves*, GTM 106, Springer-Verlag, 1986.
- [12] R. Tijdeman, *Diophantine equations and Diophantine approximations*, Number theory and applications (Banff, AB, 1988), Kluwer Acad. Publ., Dordrecht, 1989, pp. 215–243.

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY BC, CANADA V5A 1S6
E-mail address: bruin@member.ams.org